Calculer $\displaystyle\int_{\frac{1}{a}}^{a} \sin(\ln x)\ dx$.

Calculons préalablement $I = \displaystyle\int \sin(\ln x)\ dx$

Utilisons consécutivement deux intégrations par parties :

$u = \sin(\ln x)$	$u' = \dfrac{\cos(\ln x)}{x}$
$v = x$	$v' = 1$

$u = \cos(\ln x)$	$u' = \dfrac{-\sin(\ln x)}{x}$
$v = x$	$v' = 1$

On a :

$$\begin{aligned}
I &= \int \sin(\ln x)\ dx \\
&= x \cdot \sin(\ln x) - \int \cos(\ln x)\ dx \\
&= x \cdot \sin(\ln x) - \left(x \cdot \cos(\ln x) + \int \sin(\ln x)\ dx \right) \\
&= x \cdot \sin(\ln x) - x \cdot \cos(\ln x) - I \\
&= x \cdot (\sin(\ln x) - \cos(\ln x)) - I
\end{aligned}$$

Donc

$$I = \tfrac{1}{2} x \cdot (\sin(\ln x) - \cos(\ln x))$$

Finalement

$$\begin{aligned}
\int_{\frac{1}{a}}^{a} \sin(\ln x)\ dx &= \frac{a}{2}(\sin\ln a - \cos\ln a) - \frac{1}{2a}\left(\sin\ln\frac{1}{a} - \cos\ln\frac{1}{a}\right) \\
&= \frac{a}{2}(\sin\ln a - \cos\ln a) - \frac{1}{2a}(\sin(-\ln a) - \cos(-\ln a)) \\
&= \frac{a}{2}(\sin\ln a - \cos\ln a) - \frac{1}{2a}(-\sin a - \cos\ln a) \\
&= \sin\ln a \cdot \left(\frac{a}{2} + \frac{1}{2a}\right) + \cos\ln a \cdot \left(\frac{1}{2a} - \frac{a}{2}\right) \\
&= \boxed{\frac{1}{2a}\left[(a^2+1) \cdot \sin\ln a + (1-a^2) \cdot \cos\ln a\right]}
\end{aligned}$$