Calculer $\displaystyle\int_0^1 \frac{x-1}{x^2+x+1}\,dx$.

On a

$$\begin{aligned}
\int_0^1 \frac{x-1}{x^2+x+1}\,dx &= \int_0^1 \frac{\frac{1}{2}(2x+1)-\frac{3}{2}}{x^2+x+1}\,dx \\
&= \frac{1}{2}\int_0^1 \frac{2x+1}{x^2+x+1} - \frac{3}{2}\int_0^1 \frac{1}{x^2+x+1}\,dx \\
&= \frac{1}{2}\ln(x^2+x+1)\Big]_0^1 - \frac{3}{2}\int_0^1 \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \\
&= \frac{1}{2}\ln(x^2+x+1) - \frac{3}{2}\frac{1}{\sqrt{\frac{3}{4}}}\arctan\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\Bigg]_0^1 \\
&= \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}\Bigg]_0^1 \\
&= \left(\frac{1}{2}\ln 3 - \frac{1}{2}\ln 1\right) - \left(\sqrt{3}\arctan\sqrt{3} - \sqrt{3}\arctan\frac{1}{\sqrt{3}}\right) \\
&= \boxed{\frac{1}{2}\ln 3 - \frac{\pi\sqrt{3}}{6}}
\end{aligned}$$