

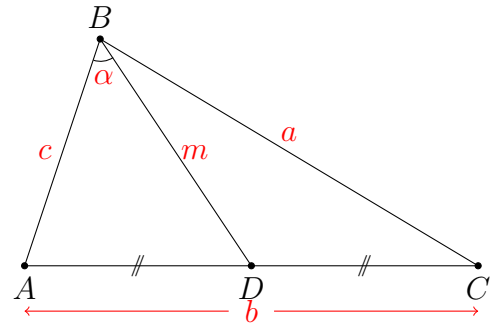
Soit D le milieu du côté $[AC]$ du triangle ABC et soit $\alpha = \widehat{ABD}$. Démontrer que si $\tan \widehat{B} = 2 \sin \widehat{A}$, alors $\alpha = \frac{\widehat{A}}{2}$.

Hypothèse : ABC est un triangle, D est le milieu de $[AC]$ et

$$\alpha = \widehat{ABD}.$$

$$\tan \widehat{B} = 2 \sin \widehat{A} \text{ donc } \sin \widehat{B} = 2 \sin \widehat{A} \cos \widehat{B} \quad (1).$$

Thèse : $\alpha = \frac{\widehat{A}}{2}$



Appelons m la longueur de la médiane $[BD]$. On a immédiatement $\widehat{BDA} = 180 - \alpha - \widehat{A}$, $\widehat{C} = 180 - \widehat{A} - \widehat{B}$ et $\widehat{BDC} = \widehat{A} + \alpha$.

- Dans le triangle BDC , la relation des sinus fournit

$$\begin{aligned} \frac{|BC|}{\sin \widehat{BDC}} &= \frac{|BD|}{\sin \widehat{C}} = \frac{|DC|}{\sin \widehat{DBC}} &\iff & \frac{|BC|}{\sin (\widehat{A} + \alpha)} = \frac{m}{\sin (\widehat{A} + \widehat{B})} = \frac{\frac{1}{2}|AC|}{\sin (\widehat{B} - \alpha)} \\ & &\iff & \frac{a}{\sin (\widehat{A} + \alpha)} = \frac{m}{\sin (\widehat{A} + \widehat{B})} = \frac{b}{2 \sin (\widehat{B} - \alpha)} \end{aligned} \quad (2)$$

- Dans le triangle ABC , la relation des sinus fournit

$$\frac{a}{\sin \widehat{A}} = \frac{b}{\sin \widehat{B}} = \frac{c}{\sin (\widehat{A} + \widehat{B})} \quad (3)$$

- Dans le triangle ABD , la relation des sinus fournit

$$\frac{m}{\sin \widehat{A}} = \frac{b}{2 \sin \alpha} = \frac{c}{\sin (\widehat{A} + \alpha)} \quad (4)$$

Partons des relations (2) : $\frac{a}{\sin (\widehat{A} + \alpha)} = \frac{m}{\sin (\widehat{A} + \widehat{B})}$

$$\begin{aligned}
\frac{a}{\sin(\widehat{A} + \alpha)} &= \frac{m}{\sin(\widehat{A} + \widehat{B})} \\
m \cdot \sin(\widehat{A} + \alpha) &= a \cdot \sin(\widehat{A} + \widehat{B}) \\
m \cdot \sin(\widehat{A} + \alpha) &= a \cdot (\sin \widehat{A} \cos \widehat{B} + \sin \widehat{B} \cos \widehat{A}) \\
m \cdot \sin(\widehat{A} + \alpha) &\stackrel{(1)}{=} a \cdot \left(\frac{\sin \widehat{B}}{2} + \sin \widehat{B} \cos \widehat{A} \right) \\
m \cdot \sin(\widehat{A} + \alpha) &= \frac{a \sin \widehat{B}}{2} \cdot (1 + 2 \cos \widehat{A}) \\
m \cdot \sin(\widehat{A} + \alpha) &\stackrel{(3)}{=} \frac{b \sin \widehat{A}}{2} \cdot (1 + 2 \cos \widehat{A}) \\
m \cdot \sin(\widehat{A} + \alpha) &\stackrel{(4)}{=} \frac{2m \sin \alpha}{2} \cdot (1 + 2 \cos \widehat{A}) \\
m \cdot \sin(\widehat{A} + \alpha) &= m \sin \alpha \cdot (1 + 2 \cos \widehat{A}) \\
\sin(\widehat{A} + \alpha) &= \sin \alpha \cdot (1 + 2 \cos \widehat{A})
\end{aligned}$$

En développant cette dernière expression, il vient

$$\begin{aligned}
\sin \widehat{A} \cos \alpha + \sin \alpha \cos \widehat{A} &= \sin \alpha + 2 \sin \alpha \cos \widehat{A} \\
\sin \widehat{A} \cos \alpha - \sin \alpha \cos \widehat{A} &= \sin \alpha \\
\sin(\widehat{A} - \alpha) &= \sin \alpha
\end{aligned}$$

On en déduit que $\alpha = \widehat{A} - \alpha$ (c'est-à-dire $\widehat{A} = 2\alpha$) ou que $\alpha = \pi - (\widehat{A} - \alpha)$ ce qui est impossible.

On en conclut que $\boxed{\frac{\widehat{A}}{2} = \alpha}$.