

**Exercices résolus de mathématiques.**

# Analyse

**ANA 1**

**EXANA010 – EXANA019**

<http://www.matheux.c.la>

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## EXANA010 – FACSA, ULG, Liège, septembre 1999

A) Rechercher l'ensemble de définition de la fonction :

$$F(x) = \int \frac{1}{e^x - 1} dx$$

B) Calculer  $F(x)$

C) En déduire la valeur de :

$$\int_1^{+\infty} \frac{1}{e^x - 1} dx$$

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A)  $e^x - 1 \neq 0 \rightarrow x \neq 0 \rightarrow \text{Domaine : } \mathbb{R} \setminus \{0\}$

B) Soit  $y = e^x \rightarrow dy = e^x dx = y dx \rightarrow dx = \frac{dy}{y}$

$$\begin{aligned} F(x) &= \int \frac{1}{e^x - 1} dx = \int \frac{dy}{y(y-1)} = -\int \frac{dy}{y} + \int \frac{dy}{y-1} = -\ln y + \ln(y-1) \\ &= \ln \frac{y-1}{y} = \ln \frac{e^x - 1}{e^x} = \ln \left( 1 - \frac{1}{e^x} \right) \end{aligned}$$

$$\text{C) } \int_1^{+\infty} \frac{1}{e^x - 1} dx = \left[ \ln \left( 1 - \frac{1}{e^x} \right) \right]_1^{+\infty} = \ln 1 - \ln \left( 1 - \frac{1}{e} \right) = -\ln \left( 1 - \frac{1}{e} \right)$$

**EXANA011 – FACSA, ULG, Liège, juillet 98.**

A) Calculer  $F(x) = \int \frac{dx}{\sqrt{x} \cos^2 \sqrt{x}}$

B) Calculer  $G(x) = \int \cos x \ln(1 + \cos x) dx$

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A)  $F(x) = \int \frac{dx}{\sqrt{x} \cos^2 \sqrt{x}}$

Soit  $t = \sqrt{x} \rightarrow dt = \frac{1}{2\sqrt{x}} dx = \frac{1}{2t} dt \rightarrow dx = 2t dt$

$$F(x) = \int \frac{2t dt}{t \cos^2 t} = 2 \int \frac{dt}{\cos^2 t} = 2 \tan t = \boxed{2 \tan \sqrt{x}}$$

B)  $G(x) = \int \cos x \ln(1 + \cos x) dx$

Par parties :  $f' = \cos x \rightarrow f = \sin x$

$$g = \ln(1 + \cos x) \rightarrow g' = -\frac{\sin x}{1 + \cos x}$$

$$G(x) = \sin x \ln(1 + \cos x) + \int \frac{\sin^2 x}{1 + \cos x} dx$$

$$= \sin x \ln(1 + \cos x) + \int \frac{1 - \cos^2 x}{1 + \cos x} dx$$

$$= \sin x \ln(1 + \cos x) + \int (1 - \cos x) dx$$

$$= \boxed{\sin x \ln(1 + \cos x) + x - \sin x}$$

## EXANA012 – FACSA, ULG, Liège, juillet 97.

Calculer  $F(t) = \int e^{2t} \ln(1+e^t) dt$

Suggestion : On effectuera un changement de variable suivi d'une primitivation par parties

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$$F(t) = \int e^{2t} \ln(1+e^t) dt$$

$$\text{Soit } e^t = x \rightarrow e^t dt = dx \rightarrow dt = \frac{dx}{x}$$

$$\rightarrow \int x^2 \ln(1+x) \frac{dx}{x} = \int x \ln(1+x) dx$$

$$f' = x \rightarrow f = \frac{x^2}{2}$$

$$g = \ln(1+x) \rightarrow g' = \frac{1}{1+x}$$

$$\begin{aligned} \rightarrow \frac{x^2}{2} \ln(1+x) - \frac{1}{2} \int \frac{x^2}{1+x} dx &= \frac{x^2}{2} \ln(1+x) - \frac{1}{2} \int (x-1) dx - \frac{1}{2} \int \frac{dx}{1+x} \\ &= \frac{x^2}{2} \ln(1+x) - \frac{(x-1)^2}{4} - \frac{1}{2} \ln(1+x) = \frac{1}{2} \left[ (x^2-1) \ln(1+x) - \frac{(x-1)^2}{2} \right] \end{aligned}$$

$$\boxed{F(t) = \frac{1}{2} \left[ (e^{2t}-1) \ln(1+e^t) - \frac{(e^t-1)^2}{2} \right]}$$

**EXANA013 – FACSA, ULG, Liège, septembre 97.**

Calculer

$$F(x) = \int \frac{3x^3 + 10x^2 - 2x}{(x^2 - 1)^2} dx$$

Décomposons en fractions rationnelles :

$$\frac{ax+b}{(x-1)^2} + \frac{cx+d}{(x+1)^2} = \frac{(a+c)x^3 + (2a+b-2c+d)x^2 + (a+2b+c-2d)x + (b+d)}{(x^2-1)^2}$$

D'où le système :

$a$	$b$	$c$	$d$	$X$	qui a pour solution	$\begin{cases} a = 4 \\ b = -\frac{5}{4} \\ c = -1 \\ d = \frac{5}{4} \end{cases}$
1		1		3		
2	1	-2	1	10		
1	2	1	-2	-2		
	1		1	0		

$$\rightarrow F(x) = \int \frac{4x - \frac{5}{4}}{(x-1)^2} dx - \int \frac{x - \frac{5}{4}}{(x+1)^2} dx$$

$$1) \int \frac{4x - \frac{5}{4}}{(x-1)^2} dx = 2 \int \frac{2x - 2 + \frac{11}{8}}{(x-1)^2} dx = 2 \int \frac{d(x-1)^2}{(x-1)^2} + \frac{11}{4} \int \frac{dx}{(x-1)^2} = 4 \ln |x-1| - \frac{11}{4(x-1)}$$

$$2) - \int \frac{x - \frac{5}{4}}{(x+1)^2} dx = -\frac{1}{2} \int \frac{2x + 2 - 2 - \frac{5}{2}}{(x+1)^2} dx = -\frac{1}{2} \int \frac{d(x+1)^2}{(x+1)^2} + \frac{9}{4} \int \frac{dx}{(x+1)^2}$$

$$= -\ln |x+1| - \frac{9}{4(x+1)}$$

$$\rightarrow F(x) = 4 \ln |x-1| - \frac{11}{4(x-1)} - \ln |x+1| - \frac{9}{4(x+1)}$$

$$= \ln \frac{(x-1)^4}{x+1} - \frac{11x+11+9x-9}{4(x-1)(x+1)}$$

$$F(x) = \ln \frac{(x-1)^4}{x+1} - \frac{5x + \frac{1}{2}}{(x-1)(x+1)} + C$$

## EXANA014 – FACSA, ULG, Liège, juillet 96.

A) Calculer  $F(x) = \int_0^{\frac{\pi}{4}} \frac{dx}{1 + \tan x}$

B) Calculer  $G(x) = \int \frac{x}{x^2 + x + 1} dx$

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A)  $t = \tan x \rightarrow x = \arctan t \rightarrow dx = \frac{dt}{1+t^2}$

Limites d'intégration :  $x = 0 \rightarrow t = 0$  et  $x = \frac{\pi}{4} \rightarrow t = 1$

$$\int_0^{\frac{\pi}{4}} \frac{dx}{1 + \tan x} = \int_0^1 \frac{dt}{(1+t)(1+t^2)}$$

Décomposons en fractions rationnelles :

$$\frac{1}{(1+t)(1+t^2)} = \frac{a}{1+t} + \frac{bt+c}{1+t^2} = \frac{(a+b)t^2 + (b+c)t + (a+c)}{(1+t)(1+t^2)}$$

$$\rightarrow \begin{cases} a+b=0 \\ b+c=0 \\ a+c=1 \end{cases} \rightarrow \begin{cases} a = \frac{1}{2} \\ b = -\frac{1}{2} \\ c = \frac{1}{2} \end{cases}$$

$$\frac{1}{(1+t)(1+t^2)} = \frac{1}{2} \left( \frac{1}{1+t} + \frac{-t+1}{1+t^2} \right) = \frac{1}{2} \left( \frac{1}{1+t} - \frac{t}{1+t^2} + \frac{1}{1+t^2} \right)$$

$$\int_0^1 \frac{dt}{(1+t)(1+t^2)} = \frac{1}{2} \left( \int_0^1 \frac{1}{1+t} dt - \int_0^1 \frac{t}{1+t^2} dt + \int_0^1 \frac{1}{1+t^2} dt \right)$$

$$= \frac{1}{2} \left( [\ln(1+t)]_0^1 - \frac{1}{2} \int_0^1 \frac{d(1+t^2)}{1+t^2} + [\arctan t]_0^1 \right)$$

$$= \frac{1}{2} \left( \ln 2 - \ln 1 - \frac{1}{2} [\ln(1+t^2)]_0^1 + \arctan 1 - \arctan 0 \right)$$

$$= \frac{1}{2} \left( \ln 2 - \frac{1}{2} \ln 2 + \frac{\pi}{4} \right) = \frac{1}{4} \left( \ln 2 + \frac{\pi}{2} \right)$$

A) Méthode alternative

$$\int_0^{\frac{\pi}{4}} \frac{1}{\tan x + 1} dx = \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos x + \sin x} dx = \frac{1}{2} \left[ \int_0^{\frac{\pi}{4}} \frac{\cos x + \sin x}{\cos x + \sin x} dx + \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x + \sin x} dx \right]$$

$$= \frac{1}{2} \left[ x + \ln |\cos x + \sin x| \right]_0^{\frac{\pi}{4}} = \frac{1}{2} \left( \frac{\pi}{4} + \ln \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \right) = \frac{1}{4} \left( \frac{\pi}{2} + \ln 2 \right)$$

Plus simple. N'est-ce pas?

$$\text{B) } G(x) = \int \frac{x}{x^2 + x + 1} dx = \frac{1}{2} \int \frac{2x + 1 - 1}{x^2 + x + 1} dx = \frac{1}{2} \int \frac{d(x^2 + x + 1)}{x^2 + x + 1} - \frac{1}{2} \int \frac{dx}{x^2 + x + 1}$$

$$= \frac{1}{2} \ln(x^2 + x + 1) - \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + 1 - \frac{1}{4}}$$

$$\text{Calculons le deuxième terme : } -\frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + 1 - \frac{1}{4}} = -\frac{1}{2} \frac{4}{3} \int \frac{dx}{\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1}$$

$$\text{Soit } \left(x + \frac{1}{2}\right) \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} x + \frac{\sqrt{3}}{3} = t \rightarrow dx = \frac{\sqrt{3}}{2} dt$$

$$\rightarrow -\frac{1}{2} \frac{4}{3} \frac{\sqrt{3}}{2} \int \frac{dt}{1 + t^2} = -\frac{\sqrt{3}}{3} \arctan t = -\frac{\sqrt{3}}{3} \arctan \left( \frac{\sqrt{3}}{2} (2x + 1) \right)$$

$$\boxed{G(x) = \frac{1}{2} \ln(x^2 + x + 1) - \frac{\sqrt{3}}{3} \arctan \left( \frac{\sqrt{3}}{2} (2x + 1) \right)}$$

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**EXANA015 – FACSA, ULG, Liège, septembre 96.**

A) Montrer que 
$$\frac{x^3+2x^2+x}{(x^3-1)^2} = \frac{x^2}{(x^3-1)^2} + \frac{x}{(x-1)^2(x^2+x+1)}$$

B) Calculer : 
$$F(x) = \int \frac{x^3+2x^2+x}{(x^3-1)^2} dx$$

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A) 
$$\begin{aligned} \frac{x^2}{(x^3-1)^2} + \frac{x}{(x-1)^2(x^2+x+1)} &= \frac{x^2}{(x-1)^2(x^2+x+1)^2} + \frac{x}{(x-1)^2(x^2+x+1)} \\ &= \frac{x^2+x(x^2+x+1)}{(x-1)^2(x^2+x+1)^2} = \frac{x^2+x^3+x^2+1}{(x-1)^2(x^2+x+1)^2} = \frac{x^3+2x^2+1}{(x^3-1)^2} \end{aligned}$$

B) 
$$F(x) = \int \frac{x^3+2x^2+x}{(x^3-1)^2} = \int \frac{x^2}{(x^3-1)^2} dx + \int \frac{x}{(x-1)^2(x^2+x+1)} dx$$

Désignons par  $A(x)$  le premier terme et par  $B(x)$  le deuxième terme.

$$A(x) = \int \frac{x^2}{(x^3-1)^2} dx = \frac{1}{3} \int \frac{d(x^3-1)}{(x^3-1)^2} dx = -\frac{1}{3(x^3-1)}$$

$$B(x) = \int \frac{x}{(x-1)^2(x^2+x+1)} dx$$

Décomposons en fractions rationnelles :

$$\frac{x}{(x-1)^2(x^2+x+1)} = \frac{ax+b}{(x-1)^2} + \frac{cx+d}{x^2+x+1}$$

On obtient le système :

$a$	$b$	$c$	$d$	$X$	qui a pour solutions
1		1		0	
1	1	-2	1	0	
1	1	1	-2	1	
	1		1	0	

$$\left\{ \begin{array}{l} a = 0 \\ b = \frac{1}{3} \\ c = 0 \\ d = -\frac{1}{3} \end{array} \right.$$

$$B(x) = \frac{1}{3} \int \frac{dx}{(x-1)^2} - \frac{1}{3} \int \frac{dx}{x^2+x+1} = \frac{1}{3} \left[ \int \frac{d(x-1)}{(x-1)^2} - \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \right]$$

$$= -\frac{1}{3(x-1)} - \frac{1}{3} \frac{4}{3} \int \frac{dx}{\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1}$$

Calculons le deuxième terme en effectuant un changement de variable:

$$t = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \rightarrow dx = \frac{\sqrt{3}}{2} dt$$

$$\rightarrow \frac{\sqrt{3}}{2} \int \frac{dt}{t^2+1} = \frac{\sqrt{3}}{2} \arctan t = \frac{\sqrt{3}}{2} \arctan \left( \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \right)$$

$$B(x) = -\frac{1}{3(x-1)} - \frac{2}{3\sqrt{3}} \arctan \left( \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \right)$$

$$F(x) = -\frac{1}{3(x^3-1)} - \frac{1}{3(x-1)} - \frac{2}{3\sqrt{3}} \arctan \left( \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3} \right)$$

$$F(x) = -\frac{1}{3} \left( \frac{x^2+x+2}{x^3+1} + \frac{2}{3\sqrt{3}} \arctan \left( \frac{\sqrt{3}}{3}(2x+1) \right) \right)$$

## EXANA016 - Polytech, UMon, questions-types 2000-2001.

Calculer

$$F(x) = \int e^{\arcsin x} dx$$

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Soit  $t = \arcsin x \Rightarrow x = \sin t$  et  $dx = \cos t dt$

$$F(x) \Rightarrow \int e^t \cos t dt$$

$$\text{Par parties : } f = e^t \Rightarrow f' = e^t$$

$$g' = \cos t \Rightarrow g = \sin t$$

$$\int e^t \cos t dt = e^t \sin t - \int e^t \sin t dt$$

$$\text{Par parties : } f = e^t \Rightarrow f' = e^t$$

$$g' = \sin t \Rightarrow g = -\cos t$$

$$\int e^t \cos t dt = e^t \sin t + e^t \cos t - \int e^t \cos t dt$$

$$\int e^t \cos t dt = \frac{e^t}{2} (\sin t + \cos t)$$

Conclusion

$$F(x) = \frac{1}{2} e^{\arcsin x} (x + \sqrt{1-x^2})$$

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**EXANA017 - POLYTECH, UMONS, Mons, questions-types  
2000-2001.**

Calculer

$$F(x) = \int x^{100} e^{-x} dx$$

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$$F(x) = \int x^{100} e^{-x} dx$$

$$f : x^{100} \rightarrow f' : 100x^{99}$$

$$g' : e^{-x} \rightarrow g : -e^{-x}$$

$$F(x) = -x^{100} e^{-x} + 100 \int x^{99} e^{-x} dx$$

$$f : x^{99} \rightarrow f' : 99x^{98}$$

$$g' : e^{-x} \rightarrow g : -e^{-x}$$

$$F(x) = -x^{100} e^{-x} + 100 \left( -x^{99} e^{-x} + 99 \int x^{98} e^{-x} dx \right)$$

Et ainsi de suite :

$$F(x) = -e^{-x} \left[ x^{100} + 100 x^{99} + 100 \cdot 99 x^{98} + 100 \cdot 99 \cdot 98 x^{97} + \dots \right]$$

$$= -e^{-x} \left[ x^{100} + 100! \left( \frac{x^{99}}{99!} + \frac{x^{98}}{98!} + \frac{x^{97}}{97!} + \dots + \frac{x}{1!} + 1 \right) \right]$$

$$= -e^{-x} \left[ x^{100} + 100! \sum_0^{99} \frac{x^n}{n!} \right]$$

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**EXANA018 - POLYTECH, UMONS, Mons, questions-types  
2000-2001.**

A) Calculer  $F(x) = \int \frac{1}{1+\sqrt{x}} dx$

B) Calculer  $F(x) = \int \frac{x}{(x+1)(x+2)(x+3)} dx$

A)  $F(x) = \int \frac{1}{1+\sqrt{x}} dx$

$y = \sqrt{x} \rightarrow x = y^2 \rightarrow dx = 2y dy$

$F(x) \rightarrow \int \frac{2y dy}{1+y} = 2 \left( -\int \frac{dy}{1+y} + \int dy \right) = 2y - \ln(y+1)$

$F(x) = 2\sqrt{x} - \ln(\sqrt{x}+1)$

B) On décompose en fractions rationnelles :

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{x+3}$$

D'où le système : 
$$\begin{cases} a+b+c=0 \\ 5a+4b+3c=1 \\ 6a+3b+2c=0 \end{cases} \rightarrow \begin{cases} a = -\frac{1}{2} \\ b = 2 \\ c = -\frac{3}{2} \end{cases}$$

$$\begin{aligned} F(x) &= -\frac{1}{2} \int \frac{dx}{x+1} + 2 \int \frac{dx}{x+2} - \frac{3}{2} \int \frac{dx}{x+3} \\ &= -\frac{1}{2} \ln(x+1) + 2 \ln(x+2) - \frac{3}{2} \ln(x+3) \end{aligned}$$

$F(x) = \ln \frac{(x+1)^2}{\sqrt{(x+1)(x+3)^3}}$

**EXANA019 - POLYTECH, UMONS, Mons, questions-types  
2000-2001.**

A) Calculer  $F(x) = \int e^{\sqrt{x}} dx$

B) Calculer  $F(x) = \int e^x \sin x dx$

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A) Soit :  $y = \sqrt{x} \rightarrow dx = 2y dy$

$$\Rightarrow F(x) = 2 \int ye^y dy$$

$$f : y \rightarrow f' : dy$$

$$g' : e^y \rightarrow g : e^y$$

$$\Rightarrow F(x) = 2 \left( ye^y - \int e^y dy \right) = 2 \left( ye^y - e^y \right) = 2e^y (y - 1)$$

$$\boxed{F(x) = 2e^{\sqrt{x}} (\sqrt{x} - 1)}$$

B)  $f : e^x \rightarrow f' : e^x$

$$g' : \sin x \rightarrow g : -\cos x$$

$$F(x) = -e^x \cos x + \int e^x \cos x dx$$

$$f : e^x \rightarrow f' : e^x$$

$$g' : \cos x \rightarrow g : \sin x$$

$$F(x) = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$\boxed{F(x) = \frac{e^x}{2} (\sin x - \cos x)}$$

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